Unbounded solutions to some reaction-diffusion-ODE systems modeling pattern formation (Shapes and other properties of solutions of PDEs)

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Unbounded solutions to some reaction-diffusion-ODE systems modeling pattern formation

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1 Introduction

The mechanism of pattern formation is one of the most interesting subjects in mathematical biology. A. M. Turing proposed a notion of the *diffusion-driven instability* in the seminal paper [13]. It means that a reaction between two chemicals with different diffusion rates may cause a destabilization of a spatially homogeneous state, thus leading to the formation of nontrivial spatial structure. This is a bifurcation that arises in a reaction-diffusion system, when there exists a spatially homogeneous stationary solution which is asymptotically stable with respect to spatially homogeneous perturbations but unstable to spatially heterogeneous perturbations. Models with the diffusion-driven instability describe a process of a destabilization of stationary spatially homogeneous steady states and evolution of the system towards spatially heterogeneous steady states.

Recently, the diffusion-driven instability has been observed in models describing a coupling of cell-localized processes with a cell-to-cell communication via diffusion. Such models are of a form of systems consisting of a single ordinary differential equation coupled with a reaction-diffusion equation:

$$u_t = f(u, v), \qquad v_t = D\Delta v + g(u, v), \tag{1.1}$$

such as in Refs. [4, 8, 10, 12]. We call the system in the form of (1.1) *reaction-diffusion-ODE system*. Simulations of different models of this type indicate a formation of dynamical, multimodal, and apparently irregular and unbounded structures, the shape of which depends strongly on initial conditions [1, 9, 10, 12].

A scalar reaction-diffusion equation (in a bounded, convex domain and the Neumann boundary conditions) cannot exhibit stable spatially heterogeneous patterns. Coupling it to an ODE fulfilling the following *autocatalysis* condition at the equilibrium $(\bar{u}, \bar{v})$

$$f_u(\bar{u}, \bar{v}) > 0 \tag{1.2}$$

leads to the diffusion-driven instability. However, in such a case, all regular Turing patterns are unstable, because the same mechanism which destabilizes constant solutions, destabilizes also all continuous spatially heterogeneous stationary solutions, [5, 6]. This instability result holds also for discontinuous patterns in case of a specific class of nonlinearities, see also [5, 6].

In this paper, we present two examples of (1.1) to understand the dynamics of non-constant solutions of the reaction-diffusion-ODE systems exhibiting the diffusion-driven instability. In both cases, we show that they have solutions which become unbounded (blow up) in a finite time.

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We begin our study by stating a result on the existence and boundedness of a solution to the initial boundary value problem for (1.1).

2 Existence of solutions

We consider the following system

$$u_t = f(u,v), \qquad \text{for} \quad x \in \overline{\Omega}, \quad t > 0, \tag{2.1}$$

$$v_t = \Delta v + g(u,v) \qquad \text{for} \quad x \in \Omega, \quad t > 0 \tag{2.2}$$

in a bounded domain $\Omega \subset \mathbb{R}^n$ for $n \geq 1$, with a C^2-boundary $\partial\Omega$, supplemented with the Neumann boundary condition

$$\partial_\nu v = 0 \qquad \text{for} \quad x \in \partial\Omega, \quad t > 0, \tag{2.3}$$

where $\partial_\nu = \frac{\partial}{\partial \nu}$ and ν denotes the unit outer normal vector to $\partial\Omega$, and with initial data

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x). \tag{2.4}$$

The nonlinearities $f = f(u,v)$ and $g = g(u,v)$ are arbitrary C^3-functions. Notice that equation (2.2) may contain an arbitrary diffusion coefficient which, however, can be rescaled and assumed to be equal to one.

Theorem 2.1 (Local-in-time solution). *Assume that* $u_0, v_0 \in L^\infty(\Omega)$. *Then, there exists* $T = T(\|u_0\|_\infty, \|v_0\|_\infty) > 0$ *such that the initial-boundary value problem* (2.1)–(2.4) *has a unique local-in-time mild solution* $u, v \in L^\infty\big([0,T],\, L^\infty(\Omega)\big)$.

We recall that a mild solution of problem (2.1)–(2.4) is a pair of measurable functions $u, v : [0,T] \times \overline{\Omega} \mapsto \mathbb{R}$ satisfying the following system of integral equations

$$u(x,t) = u_0(x) + \int_0^t f\big(u(x,s), v(x,s)\big)\, ds, \tag{2.5}$$

$$v(x,t) = e^{t\Delta} v_0(x) + \int_0^t e^{(t-s)\Delta} g\big(u(x,s), v(x,s)\big)\, ds, \tag{2.6}$$

where $e^{t\Delta}$ is the semigroup of linear operators generated by Laplacian with the Neumann boundary condition. Since our nonlinearities $f = f(u,v)$ and $g = g(u,v)$ are locally Lipschitz continuous, to construct a local-in-time unique solution of system (2.5)–(2.6), it suffices to apply the Banach fixed point theorem.

If u_0 and v_0 are more regular, *i.e.* if for some $\alpha \in (0,1)$ we have $u_0 \in C^\alpha(\overline{\Omega})$, $v_0 \in C^{2+\alpha}(\overline{\Omega})$ and $\partial_\nu v_0 = 0$ on $\partial\Omega$, then the mild solution of problem (2.1)–(2.4) is smooth and satisfies $u \in C^{1,\alpha}\big([0,T] \times \overline{\Omega}\big)$ and $v \in C^{1+\alpha/2,\,2+\alpha}\big([0,T] \times \overline{\Omega}\big)$.

3 Blowup solutions

Throughout this section, we let Ω be a bounded domain in $\mathbb{R}^n$ with a sufficiently regular boundary $\partial\Omega$. The unit outer normal vector to $\partial\Omega$ is denoted by ν, and let $\partial_\nu = \frac{\partial}{\partial\nu}$.

3.1 Resource-consumer type reaction

We consider the following system of equations

$$u_t = -au + u^p f(v), \qquad \text{for} \quad x \in \overline{\Omega}, \quad t > 0, \tag{3.1}$$

$$v_t = D\Delta v - bv - u^p f(v) + \kappa \qquad \text{for} \quad x \in \Omega, \quad t > 0, \tag{3.2}$$

where $D > 0$, $p > 1$, $a, b \in (0,\infty)$ and $\kappa \in [0,\infty)$. In equations (3.1)-(3.2), an arbitrary function $f = f(v)$ satisfies

$$f \in C^1([0,\infty)), \quad f(v) > 0 \quad \text{for} \quad v > 0, \quad \text{and} \quad f(0) = 0. \tag{3.3}$$

We supplement system (3.1)-(3.2) with the homogeneous Neumann boundary condition for v:

$$\partial_\nu v = 0 \qquad \text{for} \quad x \in \partial\Omega, \quad t > 0, \tag{3.4}$$

and with bounded, nonnegative, and continuous initial data

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x) \qquad \text{for} \quad x \in \Omega. \tag{3.5}$$

When u has a diffusion term on the right-hand side of (3.1), the model (3.1)-(3.5) can be found in literature in context of several applications. Let us mention a few of them. For $p = 2$, $f(v) = v$, and suitably chosen coefficients, we obtain either the, so-called, *Brussellator* appearing in the modeling of chemical morphogenetic processes, the *Gray-Scott model* (also known as a *model of glycolysis*, or the *Schnackenberg model.*

Nonnegative solutions to the following initial value problem for the system of ordinary differential equations:

$$\frac{d}{dt}\bar{u} = -a\bar{u} + \bar{u}^p f(\bar{v}), \qquad \frac{d}{dt}\bar{v} = -b\bar{v} - \bar{u}^p f(\bar{v}) + \kappa, \tag{3.6}$$

$$\bar{u}(0) = \bar{u}_0 \geq 0, \qquad \bar{v}(0) = \bar{v}_0 \geq 0. \tag{3.7}$$

are global-in-time and bounded on $[0,\infty)$.

A behavior of solutions the system of ODEs from (3.6) depends essentially on its parameters. Let $p=2$ and $f(v)=v$. For $a>0$ and $b>0$, this particular system has the trivial stationary nonnegative solution $(\bar{u},\bar{v})=(0,\kappa/b)$ which is an asymptotically stable solution. If, moreover, $\kappa^2>4a^2b$, we have two other nontrivial nonnegative stationary solutions which satisfy the following system of equations

$$\bar{u}=\frac{a}{\bar{v}} \qquad \text{and} \qquad -b\bar{v}-\frac{a^2}{\bar{v}}+\kappa=0.$$

Every such a constant nontrivial and *stable* solution of ODEs is an *unstable* solution of the reaction-diffusion-ODE problem (3.1)-(3.5), which means that it has the diffusion-driven instability due to the autocatalysis $f_u(\bar{u},\bar{v})=-a+2\bar{u}\bar{v}=a>0$.

We show that there are non-constant initial conditions such that the corresponding solution to the reaction-diffusion-ODE problem (3.1)-(3.5) blows up at one point and in a finite time.

Here, without loss of generality, we assume that $0\in\Omega$, where $\Omega\subset\mathbb{R}^n$ is an arbitrary bounded domain with a smooth boundary, and we rescale system (3.1)-(3.2) in such a way that the diffusion coefficient in equation (3.2) is equal to one.

In the following theorem, we prove that if u_0 is concentrated around an arbitrary point $x_0\in\Omega$ (we choose $x_0=0$, for simplicity) and if $v_0(x)=\bar{v}_0$ is a constant function, then the corresponding solution to problem (3.1)-(3.5) blows up in a finite time.

Theorem 3.1. *Assume that $f\in C^1([0,\infty))$ satisfies $\inf_{v\geq R} f(v)>0$ for each $R>0$. Let $p>1$ and $a,b,\kappa\in(0,\infty)$ be arbitrary. There exist numbers $\alpha\in(0,1)$, $\varepsilon>0$, and $R_0>0$ (depending on parameters of problem (3.1)-(3.5) and determined in the proof) such that if initial conditions $u_0,v_0\in C(\overline{\Omega})$ satisfy*

$$0<u_0(x)<\Big(u_0(0)^{1-p}+2\varepsilon^{-(p-1)}|x|^{\alpha}\Big)^{-\frac{1}{p-1}} \qquad \textit{for all} \quad x\in\Omega \tag{3.8}$$

$$u_0(0)\geq\left(\frac{a}{(1-e^{(1-p)a})F_0}\right)^{\frac{1}{p-1}}, \qquad \textit{where} \quad F_0=\inf_{v\geq R_0} f(v), \tag{3.9}$$

$$v_0(x)\equiv\bar{v}_0>R_0>0 \qquad \textit{for all} \quad x\in\Omega, \tag{3.10}$$

then the corresponding solution to problem (3.1)-(3.5) blows up at certain time $T_{max}\leq 1$. Moreover, the following uniform estimates are valid

$$0<u(x,t)<\varepsilon|x|^{-\frac{\alpha}{p-1}} \quad \textit{and} \quad v(x,t)\geq R_0 \qquad \textit{for all} \quad (x,t)\in\Omega\times[0,T_{max}). \tag{3.11}$$

Total mass $\int_\Omega \big(u(x,t)+v(x,t)\big)\,dx$ of each nonnegative solution to the reaction-diffusion problem (3.1)-(3.5) with $D\geq 0$ does not blow up, and stays uniformly bounded in $t > 0$. Indeed, we obtain that

$$\frac{d}{dt}\int_\Omega \big(u(x,t)+v(x,t)\big)\,dx = -\int_\Omega \big(au(x,t)+bv(x,t)\big)\,dx + \int_\Omega \kappa\,dx$$
$$\le -\min\{a,b\}\int_\Omega \big(u(x,t)+v(x,t)\big)\,dx + \kappa|\Omega|.$$

Theorem 3.1 shows that this *a priori* estimate is not sufficient to prevent the blow-up of solutions in a finite time.

3.1.1 Idea for proof of Theorem 3.1

We would like to give a sketch of the proof of Theorem 3.1. There are more details in [7].

It is an important key to solve the equation (3.1) with respect to $u(x,t)$, which leads to the following formula for all $(x,t) \in \Omega \times [0, T_{max})$:

$$u(x,t) = \frac{e^{-at}}{\left(\frac{1}{u_0(x)^{p-1}} - (p-1)\int_0^t f(v(x,s))e^{(1-p)as}\,ds\right)^{\frac{1}{p-1}}}. \tag{3.12}$$

Thus, it is clear that if we have an uniform lower bound for $v(x,t)$ and the initial condition satisfies (3.9), then the equation (3.12) leads to the following lower bound

$$u(x,t) \ge \frac{e^{-at}}{\left(\frac{1}{u_0(x)^{p-1}} - (1-e^{(1-p)at})a^{-1}F_0\right)^{\frac{1}{p-1}}}. \tag{3.13}$$

This implies that $u(0,t)$ blows up in finite time because the right-hand side of inequality (3.13) for $x = 0$ blows up at some $t \le 1$ under the assumption (3.9).

Therefore, it is sufficient to show the existence of a lower bound for v for all $(x,t) \in \Omega \times [0, T_{max})$ in order to finish the proof of Theorem 3.1. We have the following lemma.

Lemma 3.2. *Assume that $v(x,t)$ is a solution of the reaction-diffusion equation (3.2) with an arbitrary function $u(x,t)$ and with a constant initial condition satisfying $v_0(x) \equiv \bar{v}_0 > 0$. Suppose that there are numbers $\varepsilon > 0$ and*

$$\alpha \in \left(0, \frac{2(p-1)}{p}\right) \quad \text{if} \quad n \ge 2 \quad \text{and} \quad \alpha \in \left(0, \frac{p-1}{p}\right) \quad \text{if} \quad n = 1 \tag{3.14}$$

such that

$$0 < u(x,t) < \varepsilon|x|^{-\frac{\alpha}{p-1}} \qquad \text{for all} \quad (x,t) \in \Omega \times [0, T_{max}). \tag{3.15}$$

Then, there is an explicit number $C_0 > 0$ independent of ε such that for all $\varepsilon > 0$ we have

$$v(x,t) \ge \min\left\{\bar{v}_0, \frac{\kappa}{b}\right\} - \varepsilon^p C_0 \qquad \text{for all} \quad (x,t) \in \Omega \times [0, T_{max}). \tag{3.16}$$

Proof of Lemma 3.2. Let $z(t)$ be a solution of the problem

$$z_t = \Delta z - bz + \kappa, \quad z(x,0) = \bar{v}_0 \tag{3.17}$$

with the homogeneous Neumann boundary conditions. Then, we can rewrite equation (3.2) in the integral form

$$v(t) = z(t) - \int_0^t e^{(t-s)(\Delta - bI)}\big(u^p f(v)\big)(s)\, ds. \tag{3.18}$$

Here, we have a lower bound for $z(t)$:

$$z(t) = e^{-bt}\bar{v}_0 + \frac{\kappa}{b}\left(1 - e^{-bt}\right) \geq \min\left\{\bar{v}_0, \frac{\kappa}{b}\right\} \qquad \text{for all} \quad t \in [0, T_{max}). \tag{3.19}$$

Moreover, it is easy to see that there exists an upper bound for $v(x,t)$:

$$0 \leq v(x,t) \leq \max\left\{\|v_0\|_\infty, \frac{\kappa}{b}\right\} \equiv R_1 \qquad \text{for all} \quad (x,t) \in \Omega \times [0, T_{max}). \tag{3.20}$$

Therefore, we compute the L^∞-norm of equation (3.18) using (3.19) and (3.20), as well as the *a priori* assumption on u in (3.15) to obtain that

$$\begin{aligned} v(x,t) &\geq z(t) - \int_0^t \left\|e^{(t-s)(\Delta-b)}\big(u^p f(v)\big)(s)\right\|_\infty ds \\ &\geq \min\left\{\bar{v}_0, \frac{\kappa}{b}\right\} - \varepsilon^p C_q \left(\sup_{0\leq v\leq R_1} f(v)\right) \int_0^t \left(1 + (t-s)^{-\frac{n}{2q}}\right) \left\| |x|^{-\frac{\alpha p}{p-1}} \right\|_q ds. \end{aligned} \tag{3.21}$$

Here, we have used the following well-known estimate

$$\left\|e^{t(\Delta - bI)} w_0\right\|_\infty \leq C_q \left(1 + t^{-\frac{n}{2q}}\right) \|w_0\|_q \qquad \text{for all} \quad t > 0, \tag{3.22}$$

which is satisfied for each $w_0 \in L^q(\Omega)$, each $q \in [1, \infty]$, and with a constant $C_q = C(q, n, \Omega)$ independent of w_0 and of t.

Hence, choosing $n/2 < q < n(p-1)/(\alpha p)$ to have $n/(2q) < 1$ and $|x|^{-\frac{\alpha p}{p-1}} \in L^q(\Omega)$, we finish the proof of this lemma. □

By Lemma 3.2, the proof of Theorem 3.1 can be finished by showing an estimate (3.15). To do so, we need the following result on the Hölder continuity of $v(x,t)$.

Lemma 3.3. *Let $v(x,t)$ be a nonnegative solution of the problem*

$$v_t = \Delta v - bv - u^p f(v) + \kappa \qquad \textit{for} \quad x \in \Omega, \quad t \in [0, T_{max}) \tag{3.23}$$

$$\frac{\partial v}{\partial n} = 0 \qquad \textit{on} \quad \partial\Omega \times [0, T_{max}), \tag{3.24}$$

$$v(x,0) = \bar{v}_0 \qquad \textit{for} \quad x \in \Omega, \quad t \in [0, T_{max}), \tag{3.25}$$

where $\bar{v}_0$ is a positive constant and $u(x,t)$ is a nonnegative function. There exists a constant $\alpha \in (0,1)$ satisfying also (3.14)*, such that if the a priori estimate* (3.15) *for $u(x,t)$ holds true with some $\varepsilon > 0$, then*

$$|v(x,t) - v(y,t)| \le \varepsilon^p C|x-y|^\alpha \qquad \text{for all} \quad (x,t) \in \Omega \times [0, T_{max}),$$

where the constant $C > 0$ is independent of ε.

This lemma follows from a classical result on the Hölder continuity of solutions to the inhomogeneous heat equation.

Proof of Theorem 3.1. By assumption (3.8), we have $0 < u_0(x) < \varepsilon|x|^{-\frac{\alpha}{p-1}}$ for all $x \in \Omega$. Suppose that there exists $T_1 \in (0,1)$ such that the solution of problem (3.1)-(3.5) exists on the interval $[0, T_1]$ and satisfies

$$\sup_{x\in\Omega} |x|^{\frac{\alpha}{p-1}} u(x,t) < \varepsilon \quad \text{for all } t < T_1, \qquad \sup_{x\in\Omega} |x|^{\frac{\alpha}{p-1}} u(x,T_1) = \varepsilon. \tag{3.26}$$

First, we estimate the denominator of the fraction in (3.12) using assumption (3.8) as follows

$$\begin{aligned} \frac{1}{u_0(x)^{p-1}} &- (p-1)\int_0^t f(v(x,s))e^{(1-p)as}\,ds \\ &\ge 2\varepsilon^{1-p}|x|^\alpha + \frac{1}{u_0(0)^{p-1}} - (p-1)\int_0^t f(v(0,s))e^{(1-p)as}\,ds \\ &\quad + (p-1)\int_0^t \big(f(v(0,s)) - f(v(x,s))\big)e^{(1-p)as}\,ds. \end{aligned} \tag{3.27}$$

By the definition of T_{max} and formula (3.12), we immediately obtain

$$\frac{1}{u_0(0)^{p-1}} - (p-1)\int_0^t f(v(0,s))e^{(1-p)as}\,ds > 0 \qquad \text{for all} \quad t \in [0, T_{max}). \tag{3.28}$$

Next, we use our hypothesis (3.26) together with the Hölder continuity of $v(x,t)$ established in Lemma 3.3 to find constants $C > 0$ and $\alpha \in (0,1)$, the both independent of $\varepsilon \ge 0$, such that

$$(p-1)\int_0^t \big|f(v(0,s)) - f(v(x,s))\big|e^{(1-p)as}\,ds \le \varepsilon^p C a^{-1}|x|^\alpha \tag{3.29}$$

for all $(x,t) \in \Omega \times [0, T_1]$. Consequently, we obtain the lower bound for the denominator in (3.12)

$$\frac{1}{u_0(x)^{p-1}} - (p-1)\int_0^t f(v(x,s))e^{(1-p)as}\,ds \ge \big(2\varepsilon^{-(p-1)} - \varepsilon^p C a^{-1}\big)|x|^\alpha \tag{3.30}$$

for all $(x,t) \in \Omega \times [0,T_1]$. Finally, we choose $\varepsilon > 0$ so small that $2\varepsilon^{-(p-1)} - \varepsilon^p C a^{-1} > \varepsilon^{-(p-1)}$ and we substitute estimate (3.30) in equation (3.12) to obtain

$$0 < u(x,t) \le \frac{e^{-at}}{\left(\left(2\varepsilon^{-(p-1)} - \varepsilon^p C a^{-1}\right)|x|^\alpha\right)^{\frac{1}{p-1}}} < \frac{\varepsilon}{|x|^{\frac{\alpha}{p-1}}} \qquad \text{for all} \quad (x,t) \in \Omega \times [0,T_1].$$

This inequality for $t = T_1$ contradicts our hypothesis (3.26). $\square$

3.2 Activator-inhibitor type reaction

We consider the following initial-boundary value problem for a reaction-diffusion-ODE system:

$$u_t = -au + \frac{u^p}{v^q}, \qquad \text{for} \quad x \in \overline{\Omega}, \quad t > 0, \tag{3.31}$$

$$v_t = D\Delta v - bv + \gamma \frac{u^r}{v^s} \qquad \text{for} \quad x \in \Omega, \quad t > 0, \tag{3.32}$$

supplemented with the the initial data $u_0, v_0 \in C(\overline{\Omega})$ such that

$$u(x,0) = u_0(x) > 0, \qquad v(x,0) = v_0(x) > 0 \qquad \text{for all} \quad x \in \overline{\Omega} \tag{3.33}$$

and with the Neumann boundary conditions for v;

$$\partial_\nu v = 0 \qquad \text{for} \quad x \in \partial\Omega, \quad t > 0. \tag{3.34}$$

Here, $D > 0$, a, b, γ are nonnegative constants, and the nonlinearity exponents in (3.31)-(3.32) satisfy

$$p > 1, \qquad q > 0, \qquad r > 0, \qquad s \ge 0. \tag{3.35}$$

From the initial conditions, we have $\inf u_0 \equiv \inf_{x\in\Omega} u_0(x) > 0$ and $\inf v_0 \equiv \inf_{x\in\Omega} v_0(x) > 0$.

In the following, for simplicity of notation, we use the quantities

$$f_{0,T} \equiv \inf_{t\in[0,T]} e^{a(1-p+q)t} \qquad \text{and} \qquad g_{1,T} \equiv \sup_{t\in[0,T]} e^{b(1-r+s)t}. \tag{3.36}$$

For $p > 1$, it is easy to see that the reaction-diffusion-ODE system (3.31)-(3.32) has the diffusion-driven instability at a constant steady state. When the right-hand side of the equation (3.31) has a diffusion term and the exponents satisfy

$$0 < \frac{p-1}{r} < \frac{q}{s+1}, \tag{3.37}$$

the system (3.31)–(3.32) is an activator-inhibitor system proposed by Gierer and Meinhardt. It has been widely used to model various biological pattern formations.

First, we consider the the *kinetic system* of ordinary differential equations associated with (3.31)–(3.32):

$$\frac{d}{dt}\bar{u} = -a\bar{u} + \frac{\bar{u}^p}{\bar{v}^q}, \qquad \frac{d}{dt}\bar{v} = -b\bar{v} + \gamma\frac{\bar{u}^r}{\bar{v}^s}. \tag{3.38}$$

When $a = b = \gamma = 1$, it turns out that this dynamics already exhibits various kinds of interesting behaviors including the convergence to the equilibria $(0,0)$ and $(1,1)$, periodic solutions, unbounded oscillating global solutions, and a blowup of solutions in finite time [11]. In particular, if inequalities (3.37) and $p-1 \le r$ are satisfied, then solutions of (3.38) are global-in-time, while there are solutions blowing up in finite time under the conditions (3.37) and $p-1 > r$. Thus, our Theorem 3.4 shows that the diffusion of the inhibitor described by $v(x,t)$ *induces a blowup* of the space-inhomogeneous and non-diffusing activator $u(x,t)$ – also in the case when space-homogeneous solutions are global-in-time.

In the following, without loss of generality, we assume that $0 \in \Omega$. Moreover, system (3.31)-(3.32) is rescaled in such a way so that the diffusion coefficient in equation (3.32) is equal to one.

We prove that if u_0 is sufficiently well concentrated around an arbitrary point $x_0 \in \Omega$ (here, for simplicity of notation, we choose $x_0 = 0$), if v_0 is a constant function, and if $\gamma > 0$ is sufficiently small then the corresponding solution to problem (3.31)-(3.34) blows up in a finite time $T_{max} > 0$, *without additional restrictions* on the exponents in nonlinearities.

Theorem 3.4. *Assume the nonlinearity exponents satisfy* (3.35) *and let* $T > 0$ *be arbitrary. Suppose that* $0 \in \Omega$ *and*

- *there exists a number*

$$\alpha \in \left(0, \frac{2(p-1)}{r}\right) \quad \text{if} \quad n \ge 2 \quad \text{and} \quad \alpha \in \left(0, \frac{p-1}{r}\right) \quad \text{if} \quad n = 1$$

  such that $u_0 \in C(\overline{\Omega})$ *satisfies*

$$0 < u_0(x) \le \frac{1}{\left(u_0(0)^{1-p} + 2|x|^\alpha\right)^{\frac{1}{p-1}}} \qquad \text{for all} \quad x \in \Omega, \tag{3.39}$$

- $v(x,0) = \bar{v}_0$ *is a constant such that*

$$0 < \bar{v}_0 < R_0 \equiv \left(T(p-1)f_{0,T}(\inf_{x\in\Omega} u_0(x))^{p-1}\right)^{\frac{1}{q}} \qquad \text{for all} \quad x \in \Omega, \tag{3.40}$$

- $\gamma \in [0, \gamma_0)$, *where* $\gamma_0 = \gamma_0(u_0, \bar{v}_0, T, p, q, r, s, n)$ *is a certain number determined in the proof.*

Then the corresponding solution to problem (3.31)-(3.34) *blows up at some* $T_{max} \le T$. *Moreover, the following uniform estimates are valid*

$$0 < u(x,t) < |x|^{-\frac{\alpha}{p-1}} \qquad \text{and} \qquad 0 < v(x,t) < R_0 \tag{3.41}$$

for all $(x,t) \in \Omega \times [0, T_{max})$.

3.2.1 Idea for proof of Theorem 3.4

To show that some solutions to problems (3.31)-(3.34) blow up in a finite time, we first notice that if $\big(u(x,t), v(x,t)\big)$ is their solution, then the functions $u(x,t)e^{at}$ and $v(x,t)e^{bt}$ satisfy the following boundary-value problem

$$u_t = \frac{u^p}{v^q} f(t) \qquad \text{for} \quad x \in \overline{\Omega}, \quad t > 0, \tag{3.42}$$

$$v_t = D\Delta v + \gamma \frac{u^r}{v^s} g(t) \qquad \text{for} \quad x \in \Omega, \quad t > 0, \tag{3.43}$$

$$\partial_\nu v = 0 \qquad \text{for} \quad x \in \partial\Omega, \quad t > 0, \tag{3.44}$$

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \tag{3.45}$$

where

$$f(t) = e^{a(1-p+q)t} \qquad \text{and} \qquad g(t) = e^{b(1-r+s)t}. \tag{3.46}$$

Obviously, it suffices to prove a blowup of solutions to the new problem (3.42)-(3.45).

In the following, we would like to give a sketch of the proof of Theorem 3.4. There are more details in [2].

First, we note that, for every nonnegative $u_0, v_0 \in C(\overline{\Omega})$, $u(x,t)$ and $v(x,t)$ satisfy

$$u(x,t) \geq \inf u_0 \quad \text{and} \quad v(x,t) \geq \inf v_0 \qquad \text{for all} \quad (x,t) \in \Omega \times [0, T_{max}). \tag{3.47}$$

Now, for a given function $v(x,t)$, we solve equation (3.31) with respect to $u(x,t)$ to obtain the explicit formula for $u(x,t)$:

$$u(x,t) = \frac{1}{\left(\frac{1}{u_0(x)^{p-1}} - (p-1)\int_0^t \frac{f(\tau)}{v(x,\tau)^q}\, d\tau\right)^{\frac{1}{p-1}}}. \tag{3.48}$$

From the assumption (3.39) for $u_0(x)$, we can obtain the blowup of $u(x,t)$ at $x=0$ in finite time if $v(x,t)$ is uniformly bounded from above. Next lemma shows that an upper bound for $v(x,t)$ leads to the blowup of $u(x,t)$ in a finite time indeed.

Lemma 3.5. *Let $\big(u(x,t), v(x,t)\big)$ be a nonnegative solution to* (3.42)-(3.45) *with $\varepsilon \geq 0$ and $D > 0$. Suppose that for some constant $T > 0$ we have*

$$0 < v(x,t) < R_0 = \Big(T(p-1)f_{0,T}(\inf u_0)^{p-1}\Big)^{\frac{1}{q}} \qquad \text{for all} \quad (x,t) \in \Omega \times [0, T_{max}). \tag{3.49}$$

Then $u(x,t)$ blows up at certain $T_{max} \leq T$.

Proof of Lemma 3.5. Applying the comparison principle to equation (3.42), we obtain the estimate

$$u(x,t) \geq \bar{u}_1(t) \qquad \text{for all} \quad (x,t) \in \Omega \times [0, T_{max}), \tag{3.50}$$

where $\bar{u}_1 = \bar{u}_1(t)$ is the solution of the Cauchy problem

$$\frac{d}{dt}\bar{u}_1 = \frac{\bar{u}_1^p}{R_0^q} f_{0,T}, \qquad \bar{u}(0) = \inf u_0. \tag{3.51}$$

The function $\bar{u}_1$ may be computed explicitly:

$$\bar{u}_1(t) = \frac{1}{\left((\inf u_0)^{1-p} - t(p-1)R_0^{-q} f_{0,T}\right)^{\frac{1}{p-1}}}. \tag{3.52}$$

Recalling the definition of the number R_0 in (3.49), we obtain that $\bar{u}_1(t)$ blows up at $t = T$, which due to inequality (3.50) implies that $T_{max} \leq T$. $\square$

From Lemma 3.5, it is sufficient to obtain an upper bound for $v(x,t)$ to finish the proof of Theorem 3.4. The following lemma shows that a priori estimate for $u(x,t)$ similar to (3.15) is important to lead to the upper bound for $v(x,t)$.

Lemma 3.6. *Let $u(x,t)$ and $v(x,t)$ be a solution to problem* (3.31)-(3.33). *Suppose that there is a number*

$$\alpha \in \left(0, \frac{2(p-1)}{r}\right) \quad \text{if} \quad n \geq 2 \quad \text{and} \quad \alpha \in \left(0, \frac{p-1}{r}\right) \quad \text{if} \quad n = 1 \tag{3.53}$$

such that, a priori, the following inequality holds true

$$0 < u(x,t) < |x|^{-\frac{\alpha}{p-1}} \qquad \text{for all} \quad (x,t) \in \Omega \times [0, T_{max}). \tag{3.54}$$

Then, there is an explicit number $C_0 > 0$ such that for all $\gamma \geq 0$ we have

$$\|v(t)\|_\infty \leq \|v_0\|_\infty + \gamma C_0 \qquad \text{for all} \quad (x,t) \in \Omega \times [0, T_{max}). \tag{3.55}$$

Proof of Lemma 3.6. We use the following integral formulation of equation (3.32)

$$v(t) = e^{t\Delta} v_0 + \gamma \int_0^t e^{(t-\tau)\Delta} \left(\frac{u^r(\tau)}{v^s(\tau)} g(\tau)\right) d\tau. \tag{3.56}$$

Here, we recall the following well-known estimates for the heat semigroup which are valid for all $t > 0$, $D > 0$, and all $w_0 \in L^\infty(\Omega)$:

$$\|e^{tD\Delta} w_0\|_\infty \leq \|w_0\|_\infty \quad \text{and} \quad \|e^{tD\Delta} w_0\|_\infty \leq C_\ell \left(1 + t^{-\frac{n}{2\ell}}\right) \|w_0\|_\ell \tag{3.57}$$

for each $\ell \in [1, \infty]$, with a constant $C_\ell = C(\ell, n, D, \Omega)$ independent of w_0 and of t.

Now, we compute the L^∞-norm of equation (3.56). Using inequalities (3.57), the lower bound of v in (3.47) as well as the *a priori* assumption on u in (3.54) we obtain the estimate

$$\begin{aligned} \|v(t)\|_\infty &\leq \|v_0\|_\infty + \gamma \int_0^t \left\| e^{(t-\tau)\Delta}\left(\frac{u^r(\tau)}{v^s(\tau)} g(\tau)\right)\right\|_\infty d\tau \\ &\leq \|v_0\|_\infty + \gamma C_\ell \left(\inf v_0\right)^{-s} g_{1,T} \int_0^t \left(1+(t-\tau)^{-\frac{n}{2\ell}}\right) \left\||x|^{-\frac{\alpha r}{p-1}}\right\|_\ell d\tau, \end{aligned} \tag{3.58}$$

where the constant $g_{1,T}$ is defined in (3.36). Here, we choose $n/2 < \ell < n(p-1)/(\alpha r)$ to have $n/(2\ell) < 1$ and $|x|^{-\frac{\alpha r}{p-1}} \in L^\ell(\Omega)$ to finish the proof of lemma. □

We can show the Hölder continuity of v, which is similar to Lemma 3.3. Indeed, there exists a constant $\alpha \in (0,1)$ satisfying also (3.53) and a number $C > 0$, the both independent of $\gamma > 0$, such that

$$|v(x,t) - v(y,t)| \leq \gamma C |x-y|^\alpha \qquad \text{for all} \quad (x,t) \in \Omega \times [0, T_{max}).$$

We are ready to prove a result on the one-point blowup of solutions to the reaction-diffusion-ODE problem (3.31)-(3.34).

Proof of Theorem 3.4. Let $(u(x,t), v(x,t))$ be a solution to the problem (3.42)-(3.45). By Lemmas 3.5 and 3.6, it suffices to show the following estimate

$$0 < u(x,t) < |x|^{-\frac{\alpha}{p-1}} \qquad \text{for all} \quad (x,t) \in \Omega \times [0, T_{max}), \tag{3.59}$$

under the assumption that $\gamma > 0$ is sufficiently small. Let $T > 0$ be a number such that inequality (3.40) holds true.

By assumption (3.39), we have $0 < u_0(x) < |x|^{-\frac{\alpha}{p-1}}$ for all $x \in \Omega$, hence, by a continuity argument, inequality (3.59) is satisfied on a certain initial time interval. Suppose that there exists $T_1 \in \big(0, \min\{T_{max}, T\}\big)$ such that the solution of problem (3.42)-(3.45) exists on the interval $[0, T_1]$ and satisfies

$$\sup_{x\in\Omega} |x|^{\frac{\alpha}{p-1}} u(x,t) < 1 \quad \text{for all } t < T_1, \qquad \sup_{x\in\Omega} |x|^{\frac{\alpha}{p-1}} u(x,T_1) = 1. \tag{3.60}$$

We are going to use the explicit formula (3.48) for $u(x,t)$ and the Hölder continuity of $v(x,t)$ to obtain a contradiction with equality (3.60).

First, notice that assumption (3.39) can be written as $u_0(x)^{1-p} \geq 2|x|^\alpha + u_0(0)^{1-p}$. Thus, we may estimate the denominator of the fraction in (3.48) using this assumption as follows

$$\begin{aligned} &\frac{1}{u_0(x)^{p-1}} - (p-1)\int_0^t \frac{f(\tau)}{v(x,\tau)^q}\, d\tau \\ &\qquad \geq 2|x|^\alpha + \frac{1}{u_0(0)^{p-1}} - (p-1)\int_0^t \frac{f(\tau)}{v(0,\tau)^q}\, d\tau \\ &\qquad\quad + (p-1)\int_0^t \left(\frac{1}{v(0,\tau)^q} - \frac{1}{v(x,\tau)^q}\right) f(\tau)\, d\tau. \end{aligned} \tag{3.61}$$

By the definition of T_{max} and due to formula (3.48), we immediately obtain

$$\frac{1}{u_0(0)^{p-1}} - (p-1)\int_0^t \frac{f(\tau)}{v(0,\tau)^q}\,d\tau > 0 \qquad \text{for all} \quad t \in [0, T_{max}). \tag{3.62}$$

Next, we use our hypotheses (3.60) implying estimate (3.55) and the Hölder continuity of $v(x,t)$ as well as the lower bound of $v(x,t)$ in (3.47), to find constants $C > 0$ and $\alpha \in (0,1)$, satisfying also (3.53), such that the following inequality is satisfied:

$$(p-1)\int_0^t \left|\frac{1}{v(0,\tau)^q} - \frac{1}{v(x,\tau)^q}\right| f(\tau)\,d\tau \le \gamma C(T)|x|^\alpha \tag{3.63}$$

for all $(x,t) \in \Omega \times [0,T_1]$. Consequently, applying inequalities (3.62) and (3.63) in (3.61) we obtain the following lower bound for the denominator in (3.48)

$$\frac{1}{u_0(x)^{p-1}} - (p-1)\int_0^t \frac{f(\tau)}{v(x,\tau)^q}\,d\tau \ge \big(2-\gamma C\big)|x|^\alpha \tag{3.64}$$

for all $(x,t) \in \Omega \times [0,T_1]$. Finally, we choose $\gamma > 0$ so small that $\gamma C < 1$ (hence $2 - \gamma C > 1$) and we substitute estimate (3.64) into equation (3.52) to obtain

$$0 < u(x,t) \le \frac{1}{\Big(\big(2-\gamma C\big)|x|^\alpha\Big)^{\frac{1}{p-1}}} < \frac{1}{|x|^{\frac{\alpha}{p-1}}} \qquad \text{for all} \quad (x,t) \in \Omega \times [0,T_1].$$

This inequality for $t = T_1$ contradicts our hypothesis (3.60). □

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References

[1] S. Härting and A. Marciniak-Czochra, *Spike patterns in a reaction-diffusion-ode model with Turing instability*, Math. Methods in the Applied Sciences (2013), to appear. preprint, arXiv:1303.5362 [math.AP]

[2] G. Karch, K. Suzuki, J. Zienkiewicz, *Finite-time blowup of solutions to some activator-inhibitor systems*, to appear in DCDS-A.

[3] M. D. Li, S. H. Chen, and Y. C. Qin, Boundedness and blow up for the general activator-inhibitor model, *Acta Math. Appl. Sinica (English Ser.)*, **11** (1995), 59–68.

[4] A. Marciniak-Czochra, Receptor-based models with diffusion-driven instability for pattern formation in *Hydra. J. Biol. Sys.* **11** (2003) 293–324 .

[5] A. Marciniak-Czochra, G. Karch, & K. Suzuki, *Unstable patterns in reaction-diffusion model of early carcinogenesis*, J. Math. Pures Appl. (9) **99** (2013), 509–543

[6] A. Marciniak-Czochra, G. Karch, and K. Suzuki *Unstable patterns in autocatalytic reaction-diffusion-ODE systems*, (2013), arXiv:1301.2002 [math.AP].

[7] A. Marciniak-Czochra, G. Karch, K. Suzuki, J. Zienkiewicz, *Diffusion-driven blowup of nonnegative solutions to reaction-diffusion-ODE systems*, to appear in Differential and Integral Equations.

[8] A. Marciniak-Czochra, M. Kimmel, *Dynamics of growth and signaling along linear and surface structures in very early tumors*, Comput. Math. Methods Med. **7** (2006), 189–213.

[9] A. Marciniak-Czochra, M. Kimmel, *Modelling of early lung cancer progression: influence of growth factor production and cooperation between partially transformed cells*, Math. Models Methods Appl. Sci. **17** (2007), suppl., 1693–1719.

[10] A. Marciniak-Czochra, M. Kimmel, *Reaction-diffusion model of early carcinogenesis: the effects of influx of mutated cells*, Math. Model. Nat. Phenom. **3** (2008), 90–114.

[11] W.-M. Ni, K. Suzuki, and I. Takagi, The dynamics of a kinetic activator-inhibitor system, *J. Differential Equations*, **229** (2006), 426–465.

[12] K. Pham, A. Chauviere, H. Hatzikirou, X. Li, H.M.. Byrne, V. Cristini, J. Lowengrub, *Density-dependent quiescence in glioma invasion: instability in a simple reaction-diffusion model for the migration/proliferation dichotomy,* J. Biol. Dyn. **6** (2011) 54–71.

[13] A. M. Turing, *The chemical basis of morphogenesis*, Phil. Trans. Roy. Soc. B **237** (1952), 37–72.